Pre-lecture brain teaser (RIY)

In the following languages, three are decidable and three are undecidable. Which are which?

- $A_{CFG} = \left\{ \langle G, w \rangle \;\middle|\; G \text{ is a } CFG \text{ that generates string } w \right\}$.
- $E_{CFG} = \left\{ \langle G \rangle \;\middle|\; G \text{ is a } CFG \text{ and } L(G) = \emptyset \right\}$.
- $ALL_{CFG} = \left\{ \langle G \rangle \;\middle|\; G \text{ is a } CFG \text{ and } L(G) = \Sigma^* \right\}$.
- $A_{LBA} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } LBA \text{ that accepts string } w \right\}$.
- $E_{LBA} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } LBA \text{ where } L(M) = \emptyset \right\}$.
- $ALL_{LBA} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } LBA \text{ where } L(M) = \Sigma^* \right\}$.

ECE-374-B: Lecture 25 - Midterm 3 Review

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November 05, 2023

University of Illinois at Urbana-Champaign

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A_{CFG} decidable?

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YES!

A_{CFG} decidable? (RIY)

YES!

- $V = \{S\}$
- $T = \{0, 1\}$
- $P = \{S \to \epsilon \mid 0S0 \mid 1S1\}$
  (abbrev. for $S \to \epsilon, S \to 0S0, S \to 1S1$)

A_{CFG} decidable? (RIY)

YES!

Lemma
A CFG in Chomsky normal form can derive a string w in at most 2^n steps!

Knowing this, we can just simulate all the possible rule combinations for 2^n steps and see if any of the resulting strings matches w.

E_{CFG} decidable? (RIY)

E_{CFG} decidable? (RIY)

YES!

E_{CFG} decidable? (RIY)

YES!

In this case, we just need to know if we can get from the start variable to a string with only terminal symbols.

1. Mark all terminal symbols in G
2. Repeat until no new variables get marked:
    2.1 Mark any variable A where G has the rule $A \to U_1 U_2 \dots U_k$ where U_i is a marked terminal/variable
3. If start variable is not marked, accept. Otherwise reject.

- $V = \{S\}$
- $T = \{0, 1\}$
- $P = \{S \to \epsilon \mid 0S0 \mid 1S1\}$ (abbrev. for $S \to \epsilon, S \to 0S0, S \to 1S1$)

ALL_{CFG} decidable? (RIY)

ALL_{CFG} decidable? (RIY)

Nope!

ALL_{CFG} decidable? (RIY)

Nope!

Proof requires computation histories which are outside the scope of this course.

A_{LBA} decidable? (RIY)

A_{LBA} decidable? (RIY)

YES!

A_{LBA} decidable? (RIY)

YES!

Remember a LBA has a finite tape. Therefore we know:

1. A tape of length n where each cell can contain g symbols, you have g^n possible configurations.
2. The tape head can be in one of n positions and has q states yielding a tape that can be in qn configurations.
3. Therefore the machine can be in qng^n configurations.

A_{LBA} decidable? (RIY)

YES!

Remember a LBA has a finite tape. Therefore we know:

1. A tape of length n where each cell can contain g symbols, you have g^n possible configurations.
2. The tape head can be in one of n positions and has q states yielding a tape that can be in qn configurations.
3. Therefore the machine can be in qng^n configurations.

Lemma
If an LBA does not accept or reject in qng^n then it is stuck in a loop forever.

A_{LBA} decidable? (RIY)

Decider for A_{LBA} will do the following.

1. Simulate $\langle M \rangle$ on w for qng^n steps.
   1.1 if accepts, then accept
   1.2 if rejects, then reject
2. If neither accepts or rejects, means it's in a loop in which case, reject.

E_{LBA} decidable? (RIY)

E_{LBA} decidable? (RIY)

Nope!

E_{LBA} decidable? (RIY)

Nope!

Proof requires computational history trick, a story for another time …

ALL_{LBA} decidable? (R/Y)

ALL_{LBA} decidable? (RIY)

Nope!

ALL_{LBA} decidable? (RIY)

Nope!

No standard proof for this, but let's look at a pattern as follows.

Decidability across grammar complexities (RIY)

	DFA	*CFG*	*PDA*	*LBA*	*TM*
A	D	D	D	D	U
E	D	D	D	U	U
ALL	D	U	U	U	U

Eventually problems get too tough ...

ALL_{LBA} decidable? (RIY)

Nope!

No standard proof for this, but let's look at a pattern:

So we sort of know that ALL_{LBA} isn't decidable because we knew ALL_{CFG} wasn't (though intuition is never sufficient evidence).

Rice's theorem (RIY)

Rice's theorem: Any 'non-trivial' property about the language recognized by a Turing machine is undecidable.

Un-/decidability practice problems

Available Undecidable languages

Undecidable due to the Theorem we proved. (Anchor point)

- $L_{Accept} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and accepts } w \right\}.$
- $L_{HALT} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } TM \text{ and halts on } \varepsilon \right\}.$

Undecidable as we did: $L_{Accept} \Rightarrow L_{Halt}$

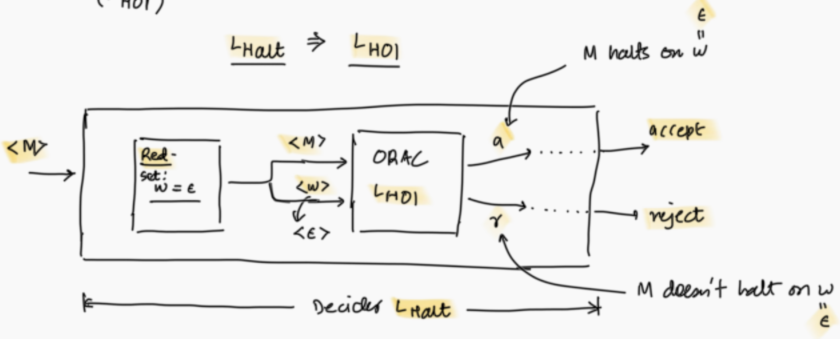
Practice 1: Halt on Input

Is the following language undecidable?

$$L_{HaltOnInput} = \left\{ \langle M, w \rangle \;\middle|\; M \text{ is a } TM \text{ and halts on } w \right\}.$$

(L_{HOI})

$\underline{L_{Halt}} \Rightarrow \underline{L_{HOI}}$



Try: $L_{Accept} \Rightarrow L_{HOI}$ (!)

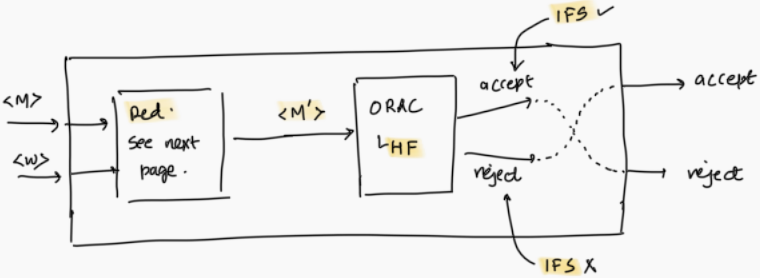
Practice 2: L has an infinite fooling set

Is the following language undecidable?

$$L_{HasFooling} = \left\{ \langle M \rangle \;\middle|\; M \text{ is a } TM \text{ and } L(M) \text{ has } \cancel{a} \text{ an infinite fooling set} \right\}.$$

(L_{HF}) (IFS)

$L_{HOI} \Rightarrow L_{HF}$



Reduction:

$\langle M \rangle, \langle w \rangle \longrightarrow \langle M' \rangle$

non-reg.

$M'(x)$:

if x is of the form $0^n 1^n$

accept   Has a Fooling set

Else:

Run M on w

accept

$\}$ if M halts on w then accept

$L(M') = \{0^n 1^n : n \geq 0\}$

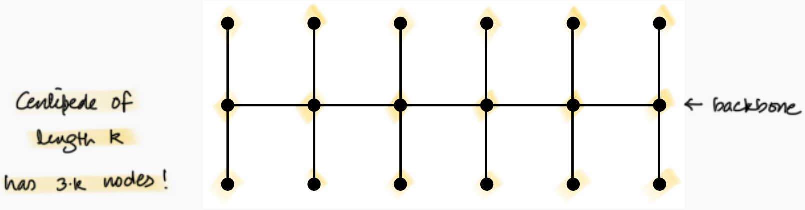
if M doesn't halt then M' accepts $0^n 1^n$: IFS ✓

if M halts then what's the language of M' : $L(M) = \Sigma^*$ — RegEx: $(0+1)^*$

doesn't have an IFS!

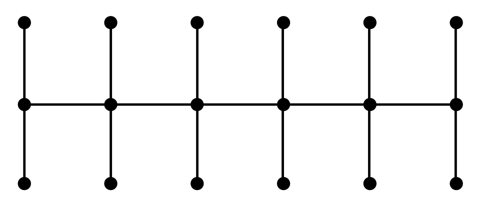
NP-Complete practice problems

A *centipede* is an undirected graph formed by a path of length k with two edges (legs) attached to each node on the path as shown in the below figure. Hence, the centipede graph has $3k$ vertices. The **CENTIPEDE** problem is the following: given an undirected graph $G = (V, E)$ and an integer k, does G contain a *centipede* of $3k$ distinct vertices as a subgraph? Prove that **CENTIPEDE** is **NP-Complete**.



Practice: NP-Complete Reduction

What do we need to do to prove Centipede is NP-Complete?

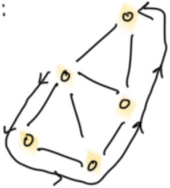


- Centipede is in NP:
  - certificate: 3 arrays of vertices: $S=\{v_1, v_2, \dots, v_k\}$: backbone
    $S'=\{v_1', v_2', \dots, v_k'\}$: upper-leg
    $S''=\{v_1'', v_2'', \dots, v_k''\}$: lower-leg
  - certifier: $|S|=k$, $|S'|=k$, $|S''|=k$
    v_i in S have edges with v_i' in S' and v_i'' in S
    poly-time!

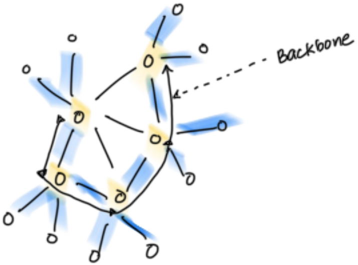
- Hardness: Hamiltonian Path ⇒ Centipede.

E.g.

G:

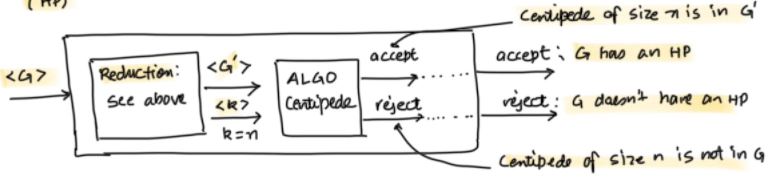


Reduction



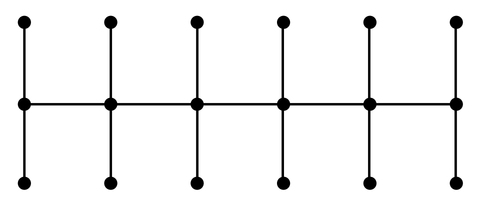
G has a Hamiltonian Path! (HP)

G' has a centipede of length k=n



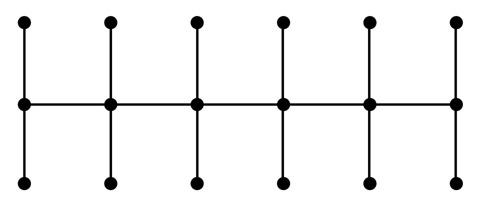
Practice: NP-Complete Reduction I

Prove Centipede is in **NP**:

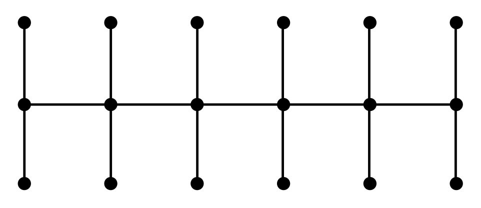


The problem is in NP. We let the certificate be three ordered lists of length k. The first list is the main path and the other two lists form the legs. We can easily verify in polynomial time that the lists form a *centipede* by checking that in the first list any two consecutive vertices have an edge between them in G and checking that a vertex from the second or third list has an edge to a vertex in the first list of the same order (position in the list).

Prove Centipede is in **NP-hard**:



Prove Centipede is in NP-hard:



Hamiltonian Path (HP): Given a graph G (either directed or undirected), is there a path that visits every vertex exactly once.

Practice: NP-Complete Reduction I

$$HP \leq_P Centipede$$

The problem is NP-Hard by reduction from HAMILTONIAN-PATH to CENTIPEDE. Given an instance of HAMILTONIAN-PATH, a graph G with n vertices $v_1, v_2, \cdots, v_n$, create a new graph G′ by adding $2n$ vertices: $u_1, u_2, \cdots, u_n$ and $x_1, x_2, \cdots, x_n$. Then, add an edge (u_i, v_i) and (x_i, v_i) for $1 \leq i \leq n$. The reduction is polynomial time since we only added $2n$ of vertices and edges to the graph.

Practice: NP-Complete Reduction II (RIY)

A **quasi-satisfying assignment (quasiSAT)** for a 3CNF boolean formula Φ is an assignment of truth values to the variables such that at most one clause in Φ does not contain a true literal. Prove that it is NP-complete to determine whether a given 3CNF boolean formula has a quasi-satisfying assignment.

quasiSAT is NP-Complete.

Practice: NP-Complete Reduction II

A **quasi-satisfying assignment (quasiSAT)** for a 3CNF boolean formula Φ is an assignment of truth values to the variables such that at most one clause in Φ does not contain a true literal. Prove that it is NP-complete to determine whether a given 3CNF boolean formula has a quasi-satisfying assignment.

Prove quasiSAT is in NP.

DIY

- certificate: ______
- certifier: ______

Practice: NP-Complete Reduction II

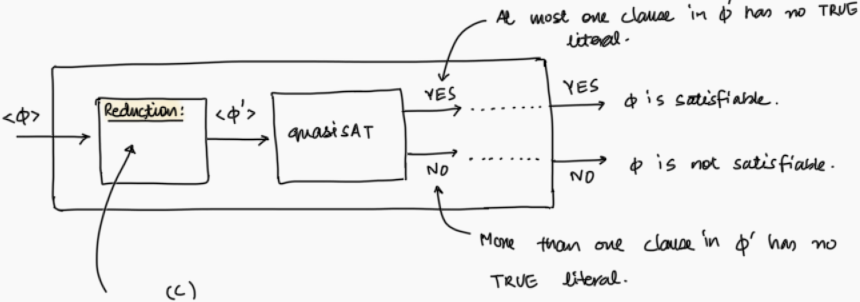
A **quasi-satisfying assignment (quasiSAT)** for a 3CNF boolean formula Φ is an assignment of truth values to the variables such that at most one clause in Φ does not contain a true literal. Prove that it is NP-complete to determine whether a given 3CNF boolean formula has a quasi-satisfying assignment.

Prove quasiSAT is NP-hard.

Practice: NP-Complete Reduction II

Prove quasiSAT is NP-hard.

3SAT $\Rightarrow$ quasiSAT



Join a clause (c) to ϕ with $\wedge$ such that c is never satisfied.

You can join a bunch of clauses to ϕ such that ϕ' (the resulting clause) is never satisfied! How?

Define: $\phi' = \phi \wedge (x \vee y \vee z) \wedge (\bar{x} \vee y \vee z) \wedge \cdots$ all 8 possibilities.

Practice: NP-Complete Reduction II

Prove quasiSAT is NP-hard.

3SAT: Given a boolean formula in conjunctive normal form, with exactly three distinct literals per clause, does the formula have a satisfying assignment.

Good luck on the exam