Pre-lecture brain teaser

Find the regular expression for the language containing all binary strings that do not contain the subsequence 111000

ECE-374-B: Lecture 3 - NFAs

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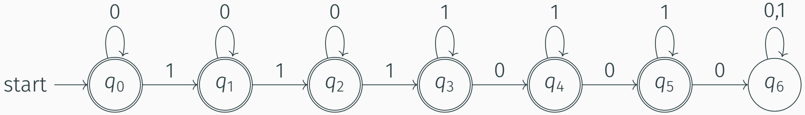
University of Illinois at Urbana-Champaign

Pre-lecture brain teaser

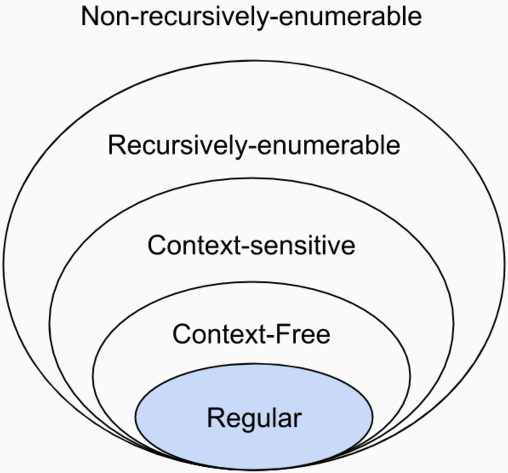
Find the regular expression for the language containing all binary strings that **do not** contain the subsequence 111000



$0^* + 0^*10^* + 0^*10^*10^* + 0^*10^*10^*1^* + 0^*10^*10^*101^* +$
$0^*10^*10^*101^*01^*$

Tangential Thought

Does luck allow us to solve unsolvable problems?



Tangential Thought

Does luck allow us to solve unsolvable problems? Consider two machines: M_1 and M_2

- M_1 is a classic deterministic machine.
- M_2 is a "lucky" machine that will always make the right choice.

Lucky machine programs

Problem: Find shortest path from a to b

Program on M_1 (Dijkstra's algorithm):

```
Initialize for each node v, Dist(s,v) = d'(s,v) = ∞
Initialize X = ∅, d'(s,s) = 0
for i = 1 to |V| do
    Let v be node realizing d'(s,v) = min_{u∈V−X} d'(s,u)
    Dist(s,v) = d'(s,v)
    X = X ∪ {v}
    Update d'(s,u) for each u in V − X as follows:
        d'(s,u) = min(d'(s,u), Dist(s,v) + ℓ(v,u))
```

Lucky machine programs

Problem: Find shortest path from a to b

Program on M_2 (Blind luck):

```
path = []
current = a
While(not at b)
    take an outgoing edge from current node
    current = new location
    path += current
return path
```

Tangential Thought

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Consider two machines: M_1 and M_2

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Question:

Tangential Thought

Does luck allow us to solve unsolvable problems?
Consider two machines: M_1 and M_2

- M_1 is a classic deterministic machine.
- M_2 is a “lucky” machine that will always make the right choice.

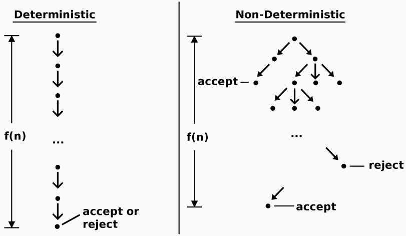
Question: Are there problems which M_2 can solve that M_1 cannot.

Non-determinism in computing

In computer science, a nondeterministic machine is a theoretical device that can have more than one output for the same input.

A machine that is capable of taking multiple states concurrently. Whenever it reaches a choice, it takes both paths.

If there is a path for the string to be accepted by the machine, then the string is part of the language.



Why non-determinism?

- Non-determinism adds power to the model; richer programming language and hence (much) easier to “design” programs
- Fundamental in **theory** to prove many theorems
- Very important in **practice** directly and indirectly
- Many deep connections to various fields in Computer Science and Mathematics

Many interpretations of non-determinism. Hard to understand at the outset. Get used to it and then you will appreciate it slowly.

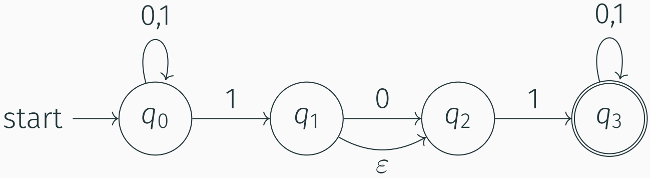
Non-deterministic finite automata (NFA) Introduction

When you come to a fork in the road, take it.

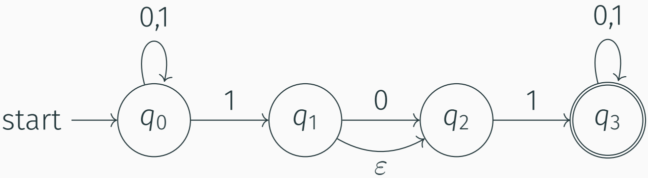
Non-deterministic Finite State Automata by example

When you come to a fork in the road, take it.

Today we'll talk about automata whose logic **is not** deterministic.



NFA acceptance: Informal



Informal definition: An NFA N accepts a string w iff some accepting state is reached by N from the start state on input w.

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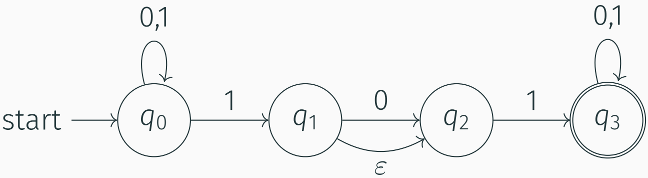
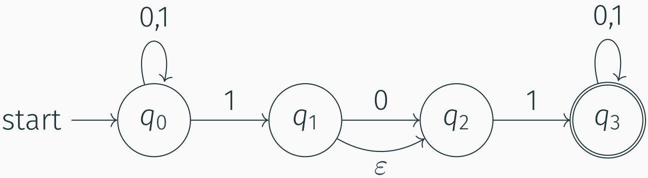


Informal definition: An NFA N accepts a string w iff some accepting state is reached by N from the start state on input w.

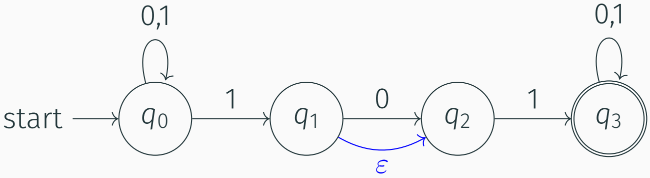
The language accepted (or recognized) by a NFA N is denote by $L(N)$ and defined as: $L(N) = \{w \mid N \text{ accepts } w\}$.

NFA acceptance: Example



- Is 010110 accepted?

NFA acceptance: Wait! what about the ϵ?!



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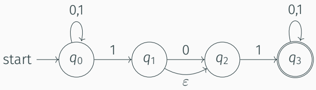

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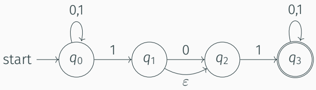

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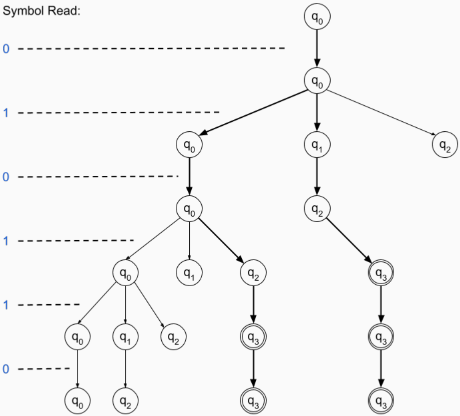

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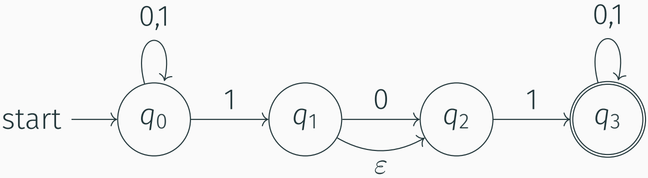
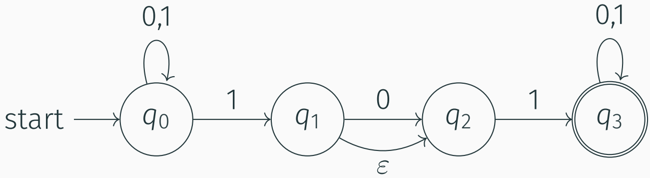


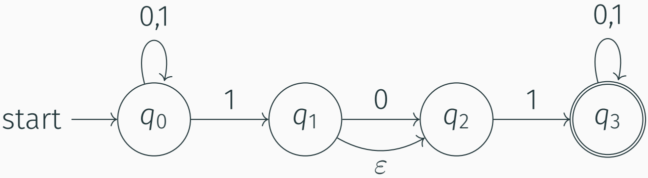
- Is 010110 accepted? **Yes**

NFA acceptance: Example



- Is 010110 accepted? **Yes**
- Is 010 accepted? **No**

NFA acceptance: Example



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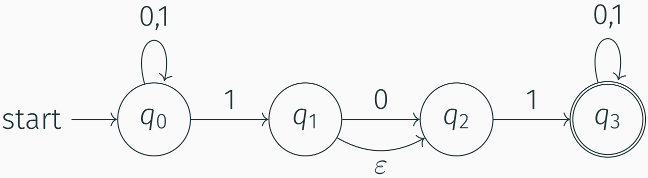
- Is 101 accepted? **Yes**

NFA acceptance: Example



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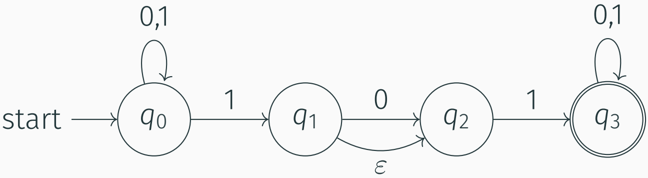

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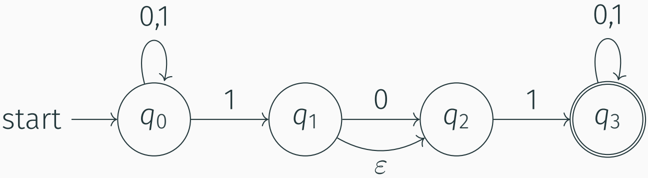

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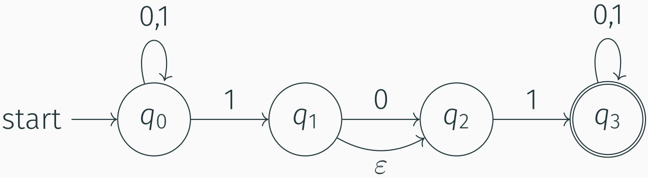

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Comment: Unlike DFAs, it is easier in NFAs to show that a string is accepted than to show that a string is **not** accepted.

Formal definition of NFA

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$\mathcal{P}(Q)$?

Reminder: Power set

Q: a set. Power set of Q is: $\mathcal{P}(Q) = 2^Q = \{X \mid X \subseteq Q\}$ is set of all subsets of Q.

Example
$Q = \{1, 2, 3, 4\}$

$$\mathcal{P}(Q) = \left\{ \begin{array}{c} \{1,2,3,4\}, \\ \{2,3,4\}, \{1,3,4\}, \{1,2,4\}, \{1,2,3\}, \\ \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \\ \{1\}, \{2\}, \{3\}, \{4\}, \\ \{\} \end{array} \right\}$$

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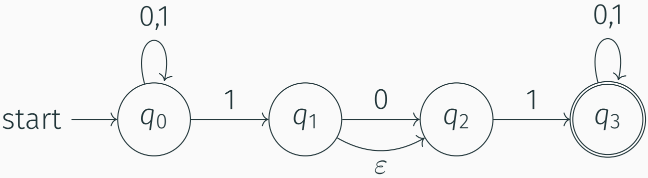
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

$\delta(q, a)$ for $a \in \Sigma \cup \{\varepsilon\}$ is a subset of Q — a set of states.

Example



- $Q = \{q_0, q_1, q_2, q_3\}$
- $\Sigma = \{0, 1\}$
- $\delta =$

	ε	0	1
q_0	$\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_1, q_2\}$	$\{q_2\}$	$\{\}$
q_2	$\{q_2\}$	$\{\}$	$\{q_3\}$
q_3	$\{q_3\}$	$\{q_3\}$	$\{q_3\}$

- $s = q_0$
- $A = \{q_3\}$

Extending the transition function to strings

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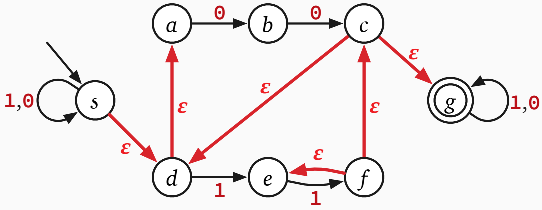
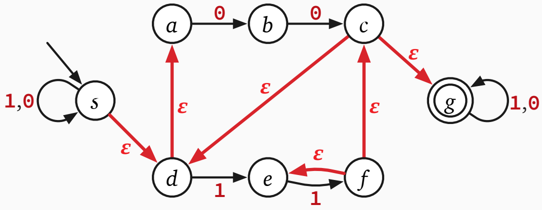
- $\delta(q, a)$: set of states that N can go to from q on reading $a \in \Sigma \cup \{\varepsilon\}$.
- Want transition function $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$
- $\delta^*(q, w)$: set of states reachable on input w starting in state q.

Definition
For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the ϵreach(q) is the set of all states that q can reach using only ε-transitions.



Definition
For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the $\epsilon\text{reach}(q)$ is the set of all states that q can reach using only ε-transitions.



Definition
For $X \subseteq Q$: $\epsilon\text{reach}(X) = \bigcup_{x \in X} \epsilon\text{reach}(x)$.

Extending the transition function to strings

ϵreach(q): set of all states that q can reach using only ε-transitions.

Definition
Inductive definition of $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$:

- if $w = \varepsilon$, $\delta^*(q, w) = \epsilon\text{reach}(q)$

Extending the transition function to strings

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Inductive definition of $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$:

- if $w = \varepsilon$, $\delta^*(q, w) = \epsilon\text{reach}(q)$
- if $w = a$ where $a \in \Sigma$:

  $$\delta^*(q, a) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q)} \delta(p, a)\right)$$

Extending the transition function to strings

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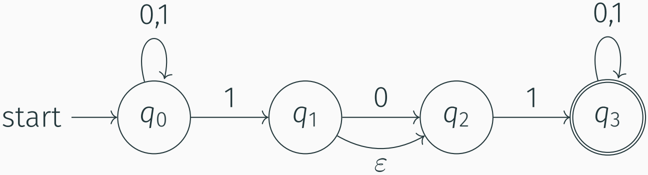
- if $w = a$ where $a \in \Sigma$:
  $$\delta^*(q, a) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q)} \delta(p, a) \right)$$
- if $w = ax$:
  $$\delta^*(q, w) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q)} \left(\bigcup_{r \in \delta^*(p,a)} \delta^*(r, x) \right) \right)$$

Example of extended transition function



Find $\delta^*(q_0, 11)$:

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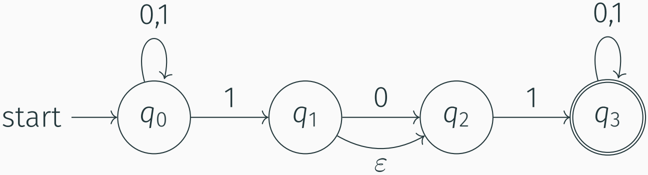

Find $\delta^*(q_0, 11)$:

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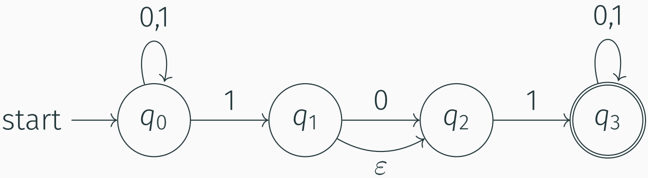

We know $w = 11 = ax$ so $a = 1$ and $x = 1$

$$\delta^*(q_0, 11) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q_0)} \left(\bigcup_{r \in \delta^*(p, 1)} \delta^*(r, 1)\right)\right)$$

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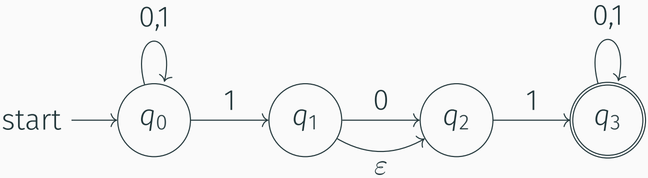

$$\epsilon\text{reach}(q_0) = \{q_0\}$$

$$\delta^*(q_0, 11) = \epsilon\text{reach}\left(\bigcup_{p \in \{q_0\}} \left(\bigcup_{r \in \delta^*(p,1)} \delta^*(r, 1)\right)\right)$$

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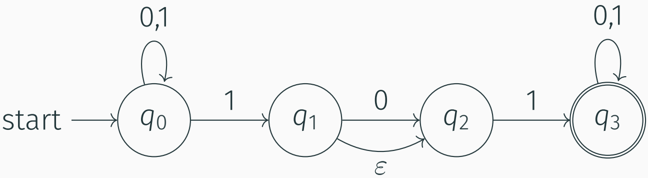

Simplify:

$$\delta^*(q_0, 11) = \epsilon\text{reach}\left(\bigcup_{r \in \delta^*(\{q_0\}, 1)} \delta^*(r, 1)\right)$$

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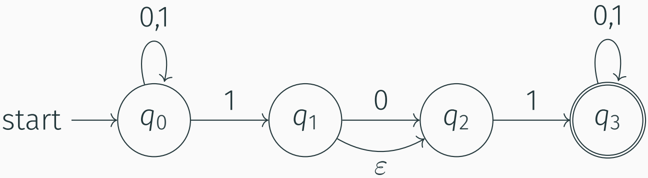

Need $\delta^*(q_0, 1) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q)} \delta(p, a)\right) = \epsilon\text{reach}(\delta\,(q_0, 1))$:

$= \epsilon\text{reach}(\{q_0, q_1\}) = \{q_0, q_1, q_2\}$

$$\delta^*(q_0, 11) = \epsilon\text{reach}\left(\bigcup_{r \in \delta^*(\{q_0\}, 1)} \delta^*(r, 1)\right)$$

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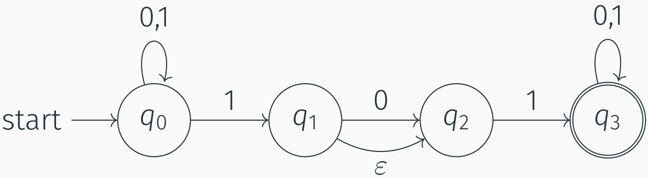

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$$\delta^*(q_0, 1) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q_0)} \delta(p, a)\right) = \epsilon\text{reach}(\delta\,(q_0, 1)):$$

$$= \epsilon\text{reach}(\{q_0, q_1\}) = \{q_0, q_1, q_2\}$$

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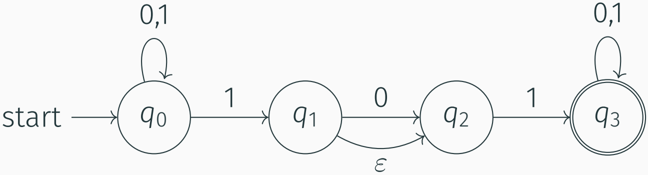

Simplify

$$\delta^*(q_0, 11) = \epsilon\text{reach}(\delta^*(q_0, 1) \cup \delta^*(q_1, 1) \cup \delta^*(q_2, 1))$$

Transition for strings: w = ax

$$\delta^*(q, w) = \epsilon\text{reach}\left(\bigcup_{p \in \epsilon\text{reach}(q)} \left(\bigcup_{r \in \delta^*(p,a)} \delta^*(r, x)\right)\right)$$

- $R = \epsilon\text{reach}(q) \implies$
  $$\delta^*(q, w) = \epsilon\text{reach}\left(\bigcup_{p \in R} \bigcup_{r \in \delta^*(p,a)} \delta^*(r, x)\right)$$
- $N = \bigcup_{p \in R} \delta^*(p, a)$: All the states reachable from q with the letter a.
- $\delta^*(q, w) = \epsilon\text{reach}\left(\bigcup_{r \in N} \delta^*(r, x)\right)$

Formal definition of language accepted by N

Definition
A string w is accepted by NFA N if $\delta_N^*(s, w) \cap A \neq \emptyset$.

Definition
The language $L(N)$ accepted by a NFA $N = (Q, \Sigma, \delta, s, A)$ is

$$\{w \in \Sigma^* \mid \delta^*(s, w) \cap A \neq \emptyset\}.$$

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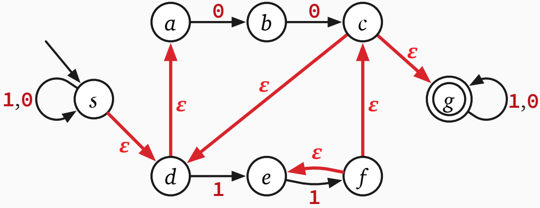
Important: Formal definition of the language of NFA above uses δ^* and not δ. As such, one does not need to include ε-transitions closure when specifying δ, since δ^* takes care of that.

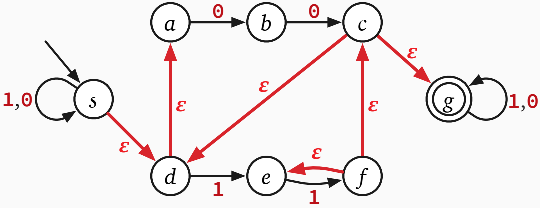
Example



What is:

- $\delta^*(s, \epsilon) = \{s, d, a\}$

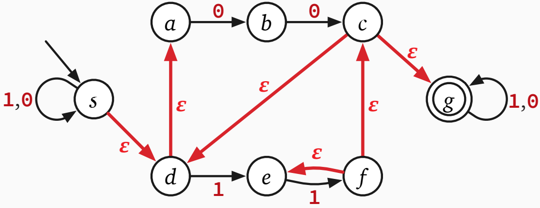
Example



What is:

- $\delta^*(s, \epsilon) = \{s, d, a\}$
- $\delta^*(s, 0) = \{s, d, a, b\}$

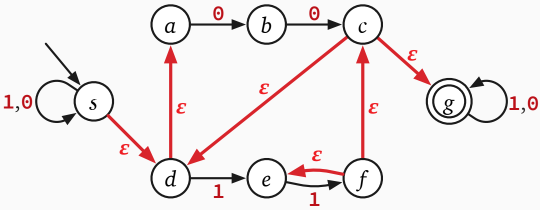
Example



What is:

- $\delta^*(s, \epsilon) = \{s, d, a\}$
- $\delta^*(s, 0) = \{s, d, a, b\}$
- $\delta^*(b, 0) = \{d, a, c, g\}$

Example



What is:

- $\delta^*(s, \epsilon) = \{s, d, a\}$
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- $\delta^*(b, 0) = \{d, a, c, g\}$
- $\delta^*(b, 00) = \{b, g\}$

Constructing generalized NFAs

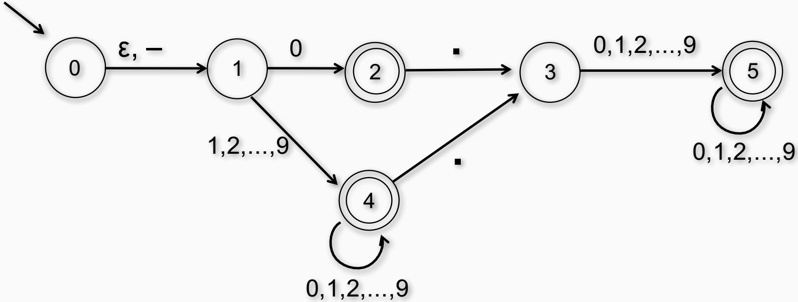
DFAs and NFAs

- Every DFA is a NFA so NFAs are at least as powerful as DFAs.
- NFAs prove ability to “guess and verify” which simplifies design and reduces number of states
- Easy proofs of some closure properties

Example

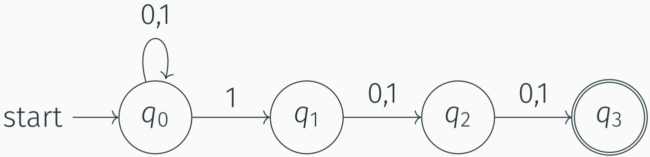
Strings that represent decimal numbers.

Examples: 154, 345.75332, 534677567.1



Example

$L = \{\text{bitstrings that have a } 1 \text{ three positions from the end}\}$



A simple transformation

Theorem
For every NFA N there is another NFA N' such that $L(N) = L(N')$ and such that N' has the following two properties:

- *N' has single final state f that has no outgoing transitions*
- *The start state s of N is different from f*

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Why couldn't we say this for DFA's?

A simple transformation

Hint: Consider the L = $0^* + 1^*$.

Closure Properties of NFAs

Closure properties of NFAs

Are the class of languages accepted by NFAs closed under the following operations?

- union
- intersection
- concatenation
- Kleene star
- complement

Closure under union

Theorem
For any two NFAs N_1 and N_2 there is a NFA N such that $L(N) = L(N_1) \cup L(N_2)$.

Closure under union

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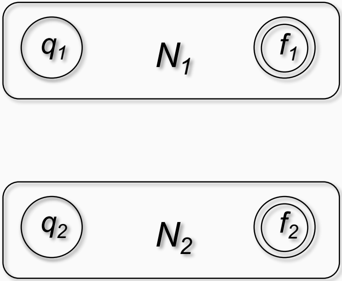

True. Single start state with ε-transition to q_1 and q_2.

Closure under concatenation

Theorem
For any two NFAs N_1 and N_2 there is a NFA N such that $L(N) = L(N_1) \bullet L(N_2)$.

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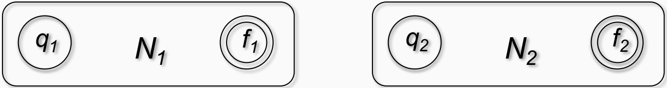
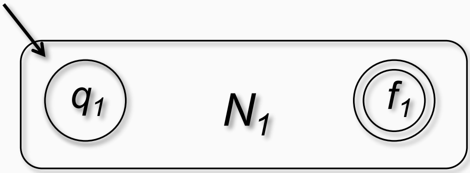


True. f_1 connected to q_2 and f_2 as accept state.

Closure under Kleene star

Theorem
For any NFA N_1 there is a NFA N such that $L(N) = (L(N_1))^*$.



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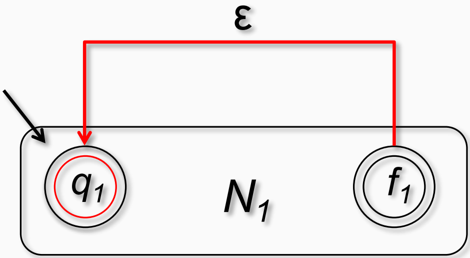

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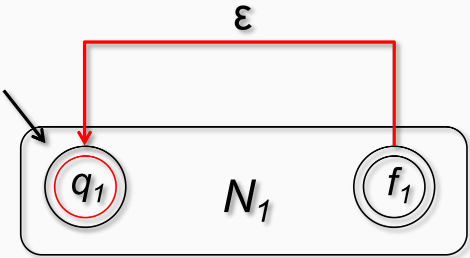
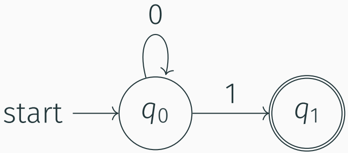


Does not work! Why?

Closure under Kleene star instructors note

Because Kleene star must include εbut if we turn the initial state into an accept state, we are inserting a εinto $L(N)$ where there might not be one. imagine:



Closure under Kleene star

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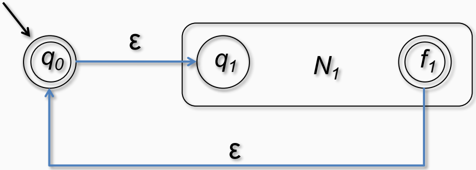
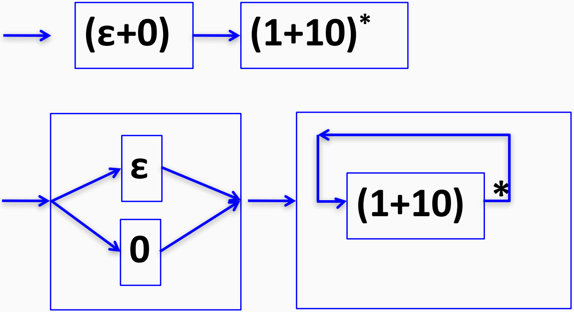


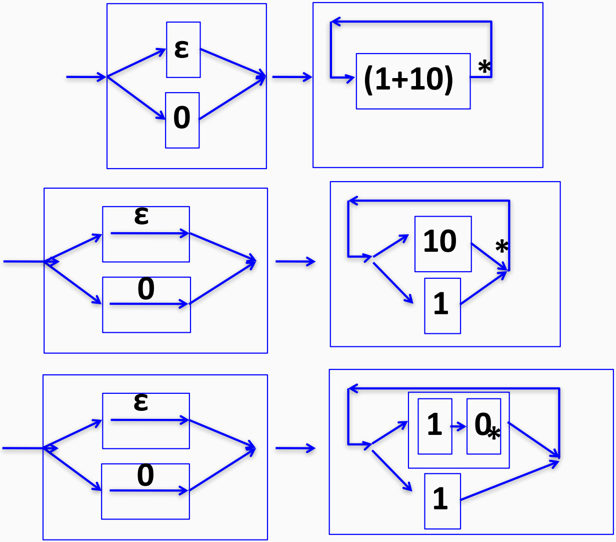
NFAs capture Regular Languages

Example

$(\varepsilon+0)(1+10)^*$

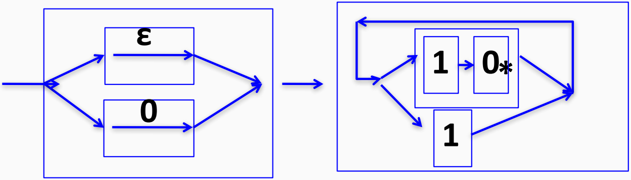


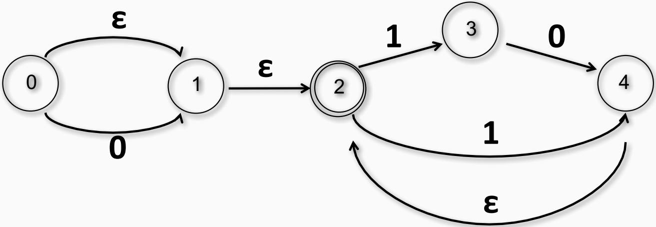
Example



Example

Final NFA simplified slightly to reduce states

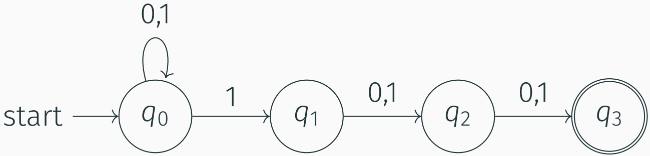




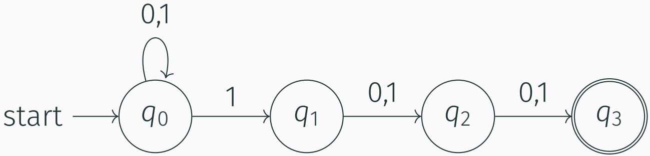
Last thought

Equivalence

Do all NFAs have a corresponding DFA?



Do all NFAs have a corresponding DFA?



Yes but it likely won't be pretty.