Pre-lecture brain teaser

Find the regular expressions for the following languages (if possible)

1. $L_1 = \{0^m 1^n | m, n \geq 0\}$ $0^* 1^*$

2. $L_2 = \{0^n 1^n \mid n \geq 0\}$ $(01)^* = \{\epsilon, 01, 0101, 010101, \ldots\}$ → doesn't work for L_2

   $= \{\epsilon, 01, 0011, 000111, \ldots\}$ "NOT POSSIBLE" Non-Regular

3. $L_3 = L_1 \cup L_2$
   - Reg. ∪ Non-Reg. (X)
   - $L_2 \subset L_1 \Rightarrow L_1 \cup L_2 = L_1 = 0^* 1^* = L_3$

4. $L_4 = L_1 \cap L_2$
   - ⇓ $L_1 \cap L_2 = L_2$: same as (2)

CS/ECE-374: Lecture 5 - Non-regularity and closure

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February 01, 2024

University of Illinois at Urbana-Champaign

Pre-lecture brain teaser

Find the regular expressions for the following languages (if possible)

1. $L_1 = \{0^m 1^n | m, n \geq 0\}$
2. $L_2 = \{0^n 1^n \mid n \geq 0\}$
3. $L_3 = L_1 \cup L_2$
4. $L_4 = L_1 \cap L_2$

Pre-lecture brain teaser

We have a language $L = \{0^n 1^n | n \geq 0\}$

Prove that L is non-regular.

Proving non-regularity: Methods

- Pumping lemma. We will not cover it but it is *sometimes* an easier proof technique to apply, but not as general as the fooling set technique.
- Closure properties. Use existing non-regular languages and regular languages to prove that some new language is non-regular.
- Fooling sets. Method of distinguishing suffixes. To prove that L is non-regular find an infinite fooling set.

Not all languages are regular

Regular Languages, DFAs, NFAs

Theorem
Languages accepted by DFAs, NFAs, and regular expressions are the same.

Question: Is every language a regular language? **No.**

Regular Languages, DFAs, NFAs

Theorem
Languages accepted by DFAs, NFAs, and regular expressions are the same.

Question: Is every language a regular language? **No.**

- Each DFA M can be represented as a string over a finite alphabet Σ by appropriate encoding
- Hence number of regular languages is countably infinite
- Number of languages is uncountably infinite
- Hence there must be a non-regular language!

A Simple and Canonical Non-regular Language

$L = \{0^n 1^n \mid n \geq 0\} = \{\epsilon, 01, 0011, 000111, \cdots, \}$

A Simple and Canonical Non-regular Language

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L is not regular.

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Intuition: Any program to recognize L seems to require counting number of zeros in input which cannot be done with fixed memory.

A Simple and Canonical Non-regular Language

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Theorem
L is not regular.

Question: Proof?

Intuition: Any program to recognize L seems to require counting number of zeros in input which cannot be done with fixed memory.

How do we formalize intuition and come up with a formal proof?

Proof by contradiction

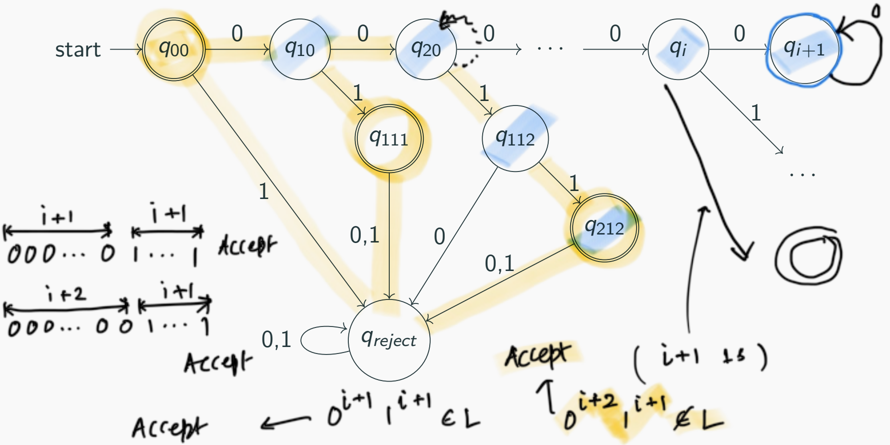
$L = \{ 0^n 1^n \mid n \geq 0 \}$

- Suppose L is regular. Then there is a DFA M such that $L(M) = L$.
- Let $M = (Q, \{0, 1\}, \delta, s, A)$ where $|Q|$ is finite.

Proof by contradiction (2)

$L = \{\epsilon, 01, 0011, 000111, \dots\}$

- Suppose L is regular. Then there is a DFA M such that $L(M) = L$.
- Let $M = (Q, \{0, 1\}, \delta, s, A)$ where $|Q|$ is finite.



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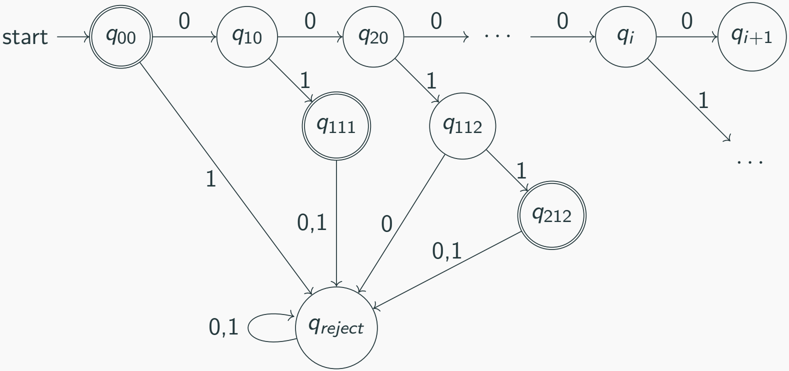

Proof by Contradiction

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Consider strings $\epsilon, 0, 00, 000, \cdots, 0^n$ total of $n + 1$ strings.

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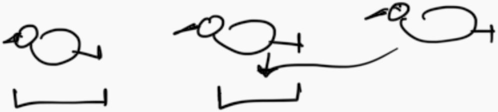
Consider strings $\epsilon, 0, 00, 000, \cdots, 0^n$ total of $n+1$ strings.

What states does M reach on the above strings? Let $q_i = \delta^*(s, 0^i)$.

By pigeon hole principle $q_i = q_j$ for some $0 \le i < j \le n$.
That is, M is in the same state after reading 0^i and 0^j where $i \neq j$.



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M should accept $0^i 1^i$ but then it will also accept $0^j 1^i$ where $i \neq j$.

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M should accept $0^i 1^i$ but then it will also accept $0^j 1^i$ where $i \neq j$.

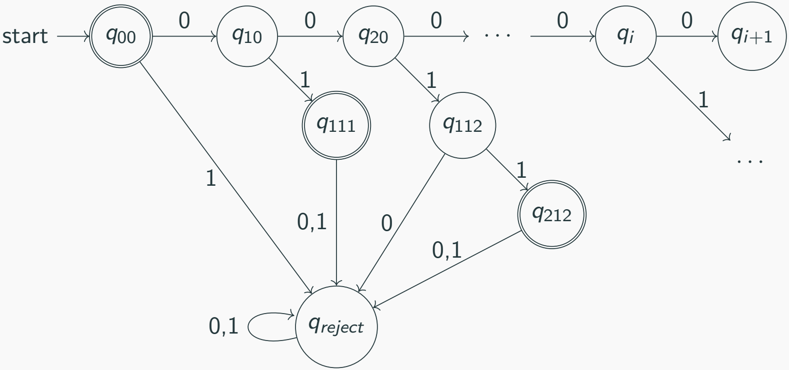
This contradicts the fact that M accepts L. Thus, there is no DFA for L.

$\Rightarrow$ L is non-regular!

$$|(0+1)^*| = \infty$$

When two states are equivalent?

States that cannot be combined?



We concluded that because each 0^i prefix has a unique state.

Are there states that aren't unique?

Can states be combined?

L_1 : Prove L_1 to be non-regular!

To show $\nexists$ a DFA $M \ni L(M) = L_1$.

finite # of states

$(Q, \Sigma, \delta, s, A)$

strategy is to show that $|Q| = \infty$!

$\Rightarrow$ M is not a DFA!

state: q

Recall: $L_2 = (0+1)^*$: M:

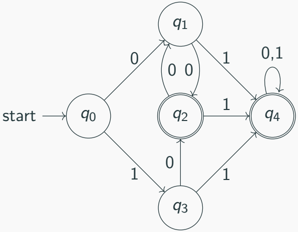
Definition
$M = (Q, \Sigma, \delta, s, A)$: DFA.

Two states $p, q \in Q$ are equivalent if for all strings $w \in \Sigma^*$, we have that

$$\delta^*(p, w) \in A \iff \delta^*(q, w) \in A.$$

One can merge any two states that are equivalent into a single state.

p, q: "equivalent"



q_1 and q_3 !

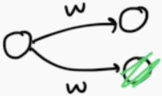
w=0

$\delta^*(q_1, 0) = q_2 \in A$

$\delta^*(q_3, 0) = q_2 \in A$ ✓

check it for all $w \in \Sigma^*$ ...

Distinguishing between states



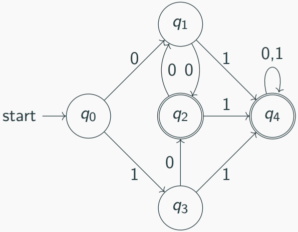
Definition
$M = (Q, \Sigma, \delta, s, A)$: DFA.

Two states $p, q \in Q$ are distinguishable if there exists a string $w \in \Sigma^*$, such that

$$\delta^*(p, w) \in A \qquad \text{and} \qquad \delta^*(q, w) \notin A.$$

or

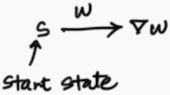
$$\delta^*(p, w) \notin A \qquad \text{and} \qquad \delta^*(q, w) \in A.$$



Distinguishable prefixes

$M = (Q, \Sigma, \delta, s, A)$: DFA

Idea: Every string $w \in \Sigma^*$ defines a state $\nabla w = \delta^*(s, w)$.



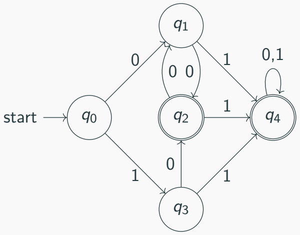
Distinguishable prefixes

$M = (Q, \Sigma, \delta, s, A)$: DFA

Idea: Every string $w \in \Sigma^*$ defines a state $\nabla w = \delta^*(s, w)$.

Definition
Two strings $u, w \in \Sigma^*$ are distinguishable for M (or $L(M)$) if ∇u and ∇w are distinguishable.

Definition (Direct restatement)
Two prefixes $u, w \in \Sigma^*$ are distinguishable for a language L if there exists a string x, such that $ux \in L$ and $wx \notin L$ (or $ux \notin L$ and $wx \in L$).



u, w ∈ Σ* Eg: u = 01 w = 11
x ∈ L Eg: x = 10

ux = 0110 ∈ L
wx = 1110 ∉ L

or vice-versa!

Distinguishable means different states

Lemma
L: regular language.

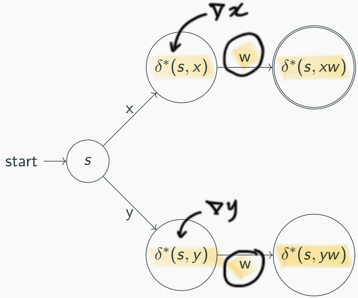
$M = (Q, \Sigma, \delta, s, A)$: DFA for L.

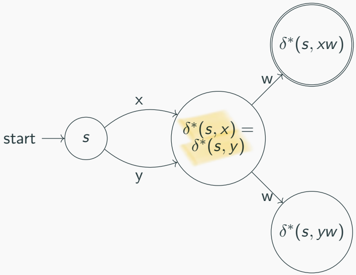
If $x, y \in \Sigma^*$ are distinguishable, then $\nabla x \neq \nabla y$. (!)

Reminder: $\nabla x = \delta^*(s, x) \in Q$ and $\nabla y = \delta^*(s, y) \in Q$

Proof by a figure

Possible	Not possible





(Think about it!)

DFA

Distinguishable strings means different states: Proof

RIY

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Proof.
Assume for the sake of contradiction that $\nabla x = \nabla y$.

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By assumption $\exists w \in \Sigma^*$ such that $\nabla xw \in A$ and $\nabla yw \notin A$.

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$\implies A \ni \nabla xw = \delta^*(s, xw) = \delta^*(\nabla x, w)$

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$= \delta^*(s, yw) = \nabla yw \notin A$.

Distinguishable strings means different states: Proof

Lemma
L: regular language.

$M = (Q, \Sigma, \delta, s, A)$: *DFA for L.*

If $x, y \in \Sigma^*$ *are distinguishable, then* $\nabla x \neq \nabla y$.

Proof.
Assume for the sake of contradiction that $\nabla x = \nabla y$.

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$= \delta^*(s, yw) = \nabla yw \notin A$.

$\implies A \ni \nabla yw \notin A$. Impossible!

Distinguishable strings means different states: Proof

Lemma
L: regular language.

$M = (Q, \Sigma, \delta, s, A)$: DFA for L.

If $x, y \in \Sigma^*$ are distinguishable, then $\nabla x \neq \nabla y$.

Proof.
Assume for the sake of contradiction that $\nabla x = \nabla y$.

By assumption $\exists w \in \Sigma^*$ such that $\nabla xw \in A$ and $\nabla yw \notin A$.

$\implies A \ni \nabla xw = \delta^*(s, xw) = \delta^*(\nabla x, w) = \delta^*(\nabla y, w)$
$= \delta^*(s, yw) = \nabla yw \notin A$.

$\implies A \ni \nabla yw \notin A$. Impossible!

Assumption that $\nabla x = \nabla y$ is false. □

Review questions...

- Prove for any $i \neq j$ then 0^i and 0^j are distinguishable for the language $\{0^n 1^n \mid n \geq 0\}$.

0^i and 0^j are distinguishable.

We need to show that $\exists\ x \in \Sigma^*$ such that

$$0^i x \in L \quad \text{but} \quad 0^j x \notin L \qquad i \neq j$$

$x = 1^i$

$$\Rightarrow \quad 0^i 1^i \in L \qquad 0^j 1^i \notin L$$

$$\Rightarrow \quad 0^i \text{ and } 0^j \text{ are distinguishable!}$$

Review questions...

- Prove for any $i \neq j$ then 0^i and 0^j are distinguishable for the language $\{0^n 1^n \mid n \geq 0\}$.
- Let L be a regular language, and let $w_1, \ldots, w_k$ be strings that are all pairwise distinguishable for L. Prove any DFA for L must have at least k states.

Review questions...

- Prove for any $i \neq j$ then 0^i and 0^j are distinguishable for the language $\{0^n1^n \mid n \geq 0\}$.
- Let L be a regular language, and let $w_1, \ldots, w_k$ be strings that are all pairwise distinguishable for L. Prove any DFA for L must have at least k states.
- Prove that $\{0^n1^n \mid n \geq 0\}$ is not regular.

By way of contradiction, let $L = \{0^n 1^n \mid n \geq 0\}$ be regular.

$\Rightarrow \exists$ a DFA $M = (Q, \Sigma, \delta, s, A)$ $|Q| =$ finite

0^i and 0^j are distinguishable for L! (✓)

$i \neq j$ 0 , 00 ; 00 , 000 ; ... } infinite FOOLING SET (infinitely many distinguishable prefixes) $\Rightarrow$ $|Q| = \infty$ $\Rightarrow$ L is not regular!

Fooling sets: Proving non-regularity

Fooling Sets

Definition
For a language L over Σ a set of strings F (could be infinite) is a fooling set or distinguishing set for L if every two distinct strings $x, y \in F$ are distinguishable.

$L = \{0^n 1^n \mid n \geq 0\}$

$F = \{0^i \mid i \geq 0\}$ is an infinite fooling set for L!

Fooling Sets

Definition
For a language L over Σ a set of strings F (could be infinite) is a fooling set or distinguishing set for L if every two distinct strings $x, y \in F$ are distinguishable.

Example: $F = \{0^i \mid i \geq 0\}$ is a fooling set for the language $L = \{0^n 1^n \mid n \geq 0\}$.

$x = 1^i$

$0^i 1^i \in L$

$0^j 1^i \notin L$

$i \neq j$

Fooling Sets

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Example: $F = \{0^i \mid i \geq 0\}$ is a fooling set for the language $L = \{0^n 1^n \mid n \geq 0\}$.

Theorem
Suppose F is a fooling set for L. If F is finite then there is no DFA M that accepts L with less than $|F|$ states.

Recall

Already proved the following lemma:

Lemma
L: regular language.

$M = (Q, \Sigma, \delta, s, A)$: DFA for L.

If $x, y \in \Sigma^*$ are distinguishable, then $\nabla x \neq \nabla y$.

Reminder: $\nabla x = \delta^*(s, x)$.

Proof of theorem

RIY

Theorem (Reworded.)
L: A language

F: a fooling set for L.

If F is finite then any DFA M that accepts L has at least $|F|$ states.

Proof.
Let $F = \{w_1, w_2, \ldots, w_m)$ be the fooling set.

Let $M = (Q, \Sigma, \delta, s, A)$ be any DFA that accepts L.

Proof of theorem

Theorem (Reworded.)
L: A language

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If F is finite then any DFA M that accepts L has at least $|F|$ states.

Proof.
Let $F = \{w_1, w_2, \ldots, w_m)$ be the fooling set.

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Let $q_i = \nabla w_i = \delta^*(s, x_i)$.

Proof of theorem

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Let $M = (Q, \Sigma, \delta, s, A)$ be any DFA that accepts L.

Let $q_i = \nabla w_i = \delta^*(s, x_i)$.

By lemma $q_i \neq q_j$ for all $i \neq j$.

As such, $|Q| \geq |\{q_1, \ldots, q_m\}| = |\{w_1, \ldots, w_m\}| = |A|$. □

Infinite Fooling Sets

Corollary
If L has an infinite fooling set F then L is not regular.

Proof.
Let $w_1, w_2, \ldots \subseteq F$ be an infinite sequence of strings such that every pair of them are distinguishable.

Assume for contradiction that $\exists$ M a DFA for L.

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If L has an infinite fooling set F then L is not regular.

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Let $F_i = \{w_1, \ldots, w_i\}$.

By theorem, $\#$ states of $M \geq |F_i| = i$, for all i.

As such, number of states in M is infinite.

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As such, number of states in M is infinite.

Contradiction: DFA = deterministic finite automata. But M not finite. $\square$

Examples

L_1 • $\{0^n1^n \mid n \geq 0\}$ $F = \{0^i \mid i \geq 0\}$ $01 \in L_1$

Non-regular! $01 \in L_2$

$10 \in L_2$

L_2 • {bitstrings with equal number of 0s and 1s}   but $10 \notin L_1$

$(L_1 \subset L_2)$ ✓ $F = \{0^i \mid i \geq 0\}$

$0^i 1^i \in L_2$ (True?)

$i \neq j$ $0^j 1^i \notin L_2$

L_3 • $\{0^k1^\ell \mid k \neq \ell\}$

Check if the same F works?

YES ✓   NO

Examples

$L = \{\text{strings of properly matched open and closing parentheses}\}$

Regular or not?

$F = \{ (^k \mid k \geq 0 \}$

Examples

$L = \{\text{palindromes over the binary alphabet} \Sigma = \{0, 1\}\}$

A palindrome is a string that is equal to its reversal, e.g. 10001 or 0110.

$F = \{ (01)^i \mid i \geq 0 \}$

$i = 1 \quad j = 2$

$(01)^i (10)^i \in L$

$0110 \in L$

$(01)^j (10)^i \notin L$

$010110 \notin L$

$\vdots$

$x = (10)^i$

$|F| = \infty$

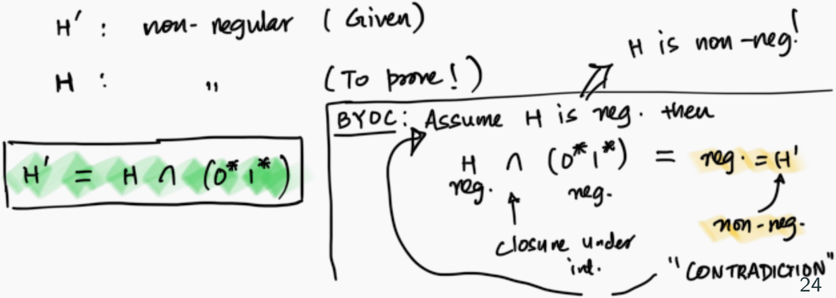
Closure properties: Proving non-regularity

Non-regularity via closure properties

$H = \{\text{bitstrings with equal number of 0s and 1s}\}$

$H' = \{0^k 1^k \mid k \geq 0\}$

Suppose we have already shown that H' is non-regular. Can we show that H is non-regular without using the fooling set argument from scratch?



Non-regularity via closure properties

$H = \{\text{bitstrings with equal number of 0s and 1s}\}$

$H' = \{0^k 1^k \mid k \geq 0\}$

Suppose we have already shown that L' is non-regular. Can we show that L is non-regular without using the fooling set argument from scratch?

$H' = H \cap L(0^* 1^*)$

Claim: The above and the fact that L' is non-regular implies L is non-regular. Why?

(We covered it in the last slide!)

Non-regularity via closure properties

$H = \{\text{bitstrings with equal number of 0s and 1s}\}$

$H' = \{0^k 1^k \mid k \geq 0\}$

Suppose we have already shown that L' is non-regular. Can we show that L is non-regular without using the fooling set argument from scratch?

$H' = H \cap L(0^*1^*)$

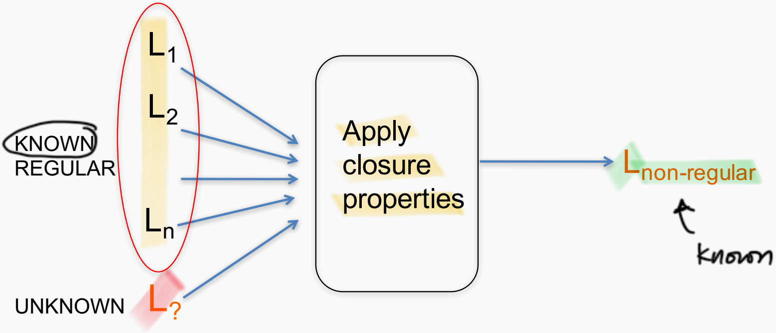
Claim: The above and the fact that L' is non-regular implies L is non-regular. Why?

Suppose H is regular. Then since $L(0^*1^*)$ is regular, and regular languages are closed under intersection, H' also would be regular. But we know H' is not regular, a contradiction.

Non-regularity via closure properties

Eg: $\{\epsilon\}$, $\{00, 10\}$, $(0+1)^*$, 0^*1^*, ...
Regular

General recipe:



Examples

$L = \{0^k 1^k \mid k \geq 1\}$

Non-Reg!

$L_1 = \{0^k 1^k \mid k \geq 0\}$: non-reg.

$L_1 = L \cup \{\epsilon\}$

L_1: non-reg.

L: non-reg.

$\{\epsilon\}$: reg.

Careful with closure!

$L' = \{0^k 1^k \mid k \geq 0\}$

Complement of L ($\overline{L}$) is also not regular.

But $L \cup \overline{L} = (0+1)^*$ which is regular.

In general, always use closure in forward direction,i.e., L and L' are regular, therefore L OP L' is regular.

In particular, regular languages are not closed under subset/superset relations.

Proving non-regularity: Summary

- Method of distinguishing suffixes. To prove that L is non-regular find an infinite fooling set.
- Closure properties. Use existing non-regular languages and regular languages to prove that some new language is non-regular.
- Pumping lemma. We did not cover it but it is sometimes an easier proof technique to apply, but not as general as the fooling set technique.