Pre-lecture brain teaser

Find the regular expressions for the following languages (if possible)

1. $L_1 = \{0^m1^n | m, n \geq 0\}$
2. $L_2 = \{0^n1^n | n \geq 0\}$
   
   $L_2$ does not work for $L_2$
   
   $\{0^n1^n | n \geq 0\} = \{\varepsilon, 01, 0101, 010101, \ldots\}$
   
   “NOT POSSIBLE” Non-Regular

3. $L_3 = L_1 \cup L_2$
   
   Reg. $\cup$ Non-Reg. $\times$

4. $L_4 = L_1 \cap L_2$
   
   $L_2 \subseteq L_1$ implies $L_1 \cup L_2 = L_1 = 0^*1^* = L_3$

   $L_1 \cap L_2 = L_2$; same as (2)
CS/ECE-374: Lecture 5 - Non-regularity and closure

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1. \( L_1 = \{0^m1^n | m, n \geq 0\} \)

2. \( L_2 = \{0^n1^n | n \geq 0\} \)

3. \( L_3 = L_1 \cup L_2 \)

4. \( L_4 = L_1 \cap L_2 \)
Pre-lecture brain teaser

We have a language $L = \{0^n1^n | n \geq 0\}$

Prove that $L$ is non-regular.
Proving non-regularity: Methods

- **Pumping lemma.** We will not cover it but it is sometimes an easier proof technique to apply, but not as general as the fooling set technique.

- **Closure properties.** Use existing non-regular languages and regular languages to prove that some new language is non-regular.

- **Fooling sets.** Method of distinguishing suffixes. To prove that $L$ is non-regular find an infinite fooling set.
Not all languages are regular
Theorem
Languages accepted by DFAs, NFA, and regular expressions are the same.

Question: Is every language a regular language? No.
Theorem
Languages accepted by DFA, NFA, and regular expressions are the same.

Question: Is every language a regular language? No.

- Each DFA $M$ can be represented as a string over a finite alphabet $\Sigma$ by appropriate encoding
- Hence number of regular languages is countably infinite
- Number of languages is uncountably infinite
- Hence there must be a non-regular language!
A Simple and Canonical Non-regular Language

\[ L = \{0^n1^n \mid n \geq 0\} = \{\varepsilon, 01, 0011, 000111, \ldots\} \]

Theorem

\[ L \text{ is not regular.} \]

Question:

Proof?

Intuition:

Any program to recognize \( L \) seems to require counting the number of zeros in input which cannot be done with fixed memory.

How do we formalize intuition and come up with a formal proof?
A Simple and Canonical Non-regular Language

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$L$ is not regular.

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How do we formalize intuition and come up with a formal proof?
Proof by contradiction

$L = \{ 0^n 1^n \mid n \geq 0 \}$

• Suppose $L$ is regular. Then there is a DFA $M$ such that $L(M) = L$.

• Let $M = (Q, \{0, 1\}, \delta, s, A)$ where $|Q|$ is finite.
Proof by contradiction

Let \( L = \{ \varepsilon, 01, 0011, 000111, \ldots \} \)

- Suppose \( L \) is regular. Then there is a DFA \( M \) such that \( L(M) = L \).
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- Suppose $L$ is regular. Then there is a DFA $M$ such that $L(M) = L$.

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Proof by Contradiction

- Suppose \( L \) is regular. Then there is a DFA \( M \) such that \( L(M) = L \).
- Let \( M = (Q, \{0, 1\}, \delta, s, A) \) where \(|Q| = n|\).
Proof by Contradiction

• Suppose $L$ is regular. Then there is a DFA $M$ such that $L(M) = L$.
• Let $M = (Q, \{0, 1\}, \delta, s, A)$ where $|Q| = n$.

Consider strings $\epsilon, 0, 00, 000, \ldots, 0^n$ total of $n + 1$ strings.
Proof by Contradiction

- Suppose $L$ is regular. Then there is a DFA $M$ such that $L(M) = L$.
- Let $M = (Q, \{0, 1\}, \delta, s, A)$ where $|Q| = n$.

Consider strings $\epsilon, 0, 00, 000, \cdots, 0^n$ total of $n+1$ strings.

What states does $M$ reach on the above strings? Let $q_i = \delta^*(s, 0^i)$.

By pigeon hole principle $q_i = q_j$ for some $0 \leq i < j \leq n$.

That is, $M$ is in the same state after reading $0^i$ and $0^j$ where $i \neq j$. 
• Suppose $L$ is regular. Then there is a DFA $M$ such that $L(M) = L$.
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$M$ should accept $0^i1^i$ but then it will also accept $0^j1^i$ where $i \neq j$. 

Proof by Contradiction

- Suppose $L$ is regular. Then there is a **DFA** $M$ such that $L(M) = L$.
- Let $M = (Q, \{0, 1\}, \delta, s, A)$ where $|Q| = n$.

Consider strings $\epsilon, 0, 00, 000, \cdots, 0^n$ total of $n + 1$ strings.

What states does $M$ reach on the above strings? Let $q_i = \delta^*(s, 0^i)$. By pigeon hole principle $q_i = q_j$ for some $0 \leq i < j \leq n$.

That is, $M$ is in the same state after reading $0^i$ and $0^j$ where $i \neq j$.

$M$ should accept $0^i1^i$ but then it will also accept $0^j1^i$ where $i \neq j$. This contradicts the fact that $M$ accepts $L$. Thus, there is no **DFA** for $L$.

$\Rightarrow L$ is non-regular!
When two states are equivalent?
We concluded that because each $0^i$ prefix has a unique state. Are there states that aren't unique? Can states be combined?
$L_1$: Prove $L_1$ to be non-regular!

To show $\not\exists$ a DFA $M \ni L(M) = L_1$.

\[
\uparrow \quad \text{finite \# of states}
\]

$(Q, \Sigma, \delta, s, A)$

strategy is to show that $|Q| = \infty$!

$\Rightarrow M$ is not a DFA!

Recall: $L_2 = \{0+1\}^*$

\[
\begin{array}{c}
\text{state: } q_1 \\
\text{M: }
\end{array}
\]
**Equivalence between states**

**Definition**

\[ M = (Q, \Sigma, \delta, s, A) : \text{DFA.} \]

Two states \( p, q \in Q \) are **equivalent** if for all strings \( w \in \Sigma^* \), we have that

\[ \delta^* (p, w) \in A \iff \delta^* (q, w) \in A. \]

One can merge any two states that are equivalent into a single state.

\( p, q : \text{"equivalent"} \)

\[ s^*(q_1, 0) = q_2 \in A \]

\[ s^*(q_3, 0) = q_2 \in A \]

Check it for all \( w \in \Sigma^* \).
**Distinguishing between states**

**Definition**

\( M = (Q, \Sigma, \delta, s, A) \): DFA.

Two states \( p, q \in Q \) are **distinguishable** if there exists a string \( w \in \Sigma^* \), such that

\[
\delta^*(p, w) \in A \quad \text{and} \quad \delta^*(q, w) \notin A.
\]

or

\[
\delta^*(p, w) \notin A \quad \text{and} \quad \delta^*(q, w) \in A.
\]
$M = (Q, \Sigma, \delta, s, A)$: DFA

**Idea:** Every string $w \in \Sigma^*$ defines a state $\nabla w = \delta^*(s, w)$. 

$$s \xrightarrow{w} \nabla w$$

Start state
Distinguishable prefixes

\[ M = (Q, \Sigma, \delta, s, A): \text{DFA} \]

**Idea:** Every string \( w \in \Sigma^* \) defines a state \( \nabla w = \delta^*(s, w) \).

**Definition**
Two strings \( u, w \in \Sigma^* \) are distinguishable for \( M \) (or \( L(M) \)) if \( \nabla u \) and \( \nabla w \) are distinguishable.

**Definition (Direct restatement)**
Two prefixes \( u, w \in \Sigma^* \) are distinguishable for a language \( L \) if there exists a string \( x \), such that \( ux \in L \) and \( wx \notin L \) (or \( ux \notin L \) and \( wx \in L \)).

\[ u, w \in \Sigma^* \text{ eg: } u = 01 \quad w = 11 \]
\[ x \in L \text{ eg: } x = 10 \]
\[ ux = 0110 \in L \]
\[ wx = 1110 \notin L \]

or vice-versa!
Distinguishable means different states

**Lemma**

$L$: regular language.

$M = (Q, \Sigma, \delta, s, A)$: DFA for $L$.

If $x, y \in \Sigma^*$ are distinguishable, then $\nabla x \neq \nabla y$. (!)

Reminder: $\nabla x = \delta^*(s, x) \in Q$ and $\nabla y = \delta^*(s, y) \in Q$
Proof by a figure

Possible

\[ \delta^*(s, x) \xrightarrow{w} \delta^*(s, xw) \]
\[ \delta^*(s, y) \xrightarrow{w} \delta^*(s, yw) \]

Not possible

\[ \delta^*(s, x) = \delta^*(s, y) \]

DFA

(Think about it!)
Distinguishable strings means different states: Proof

Lemma

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Proof.

Assume for the sake of contradiction that $\nabla x = \nabla y$. 

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Proof.
Assume for the sake of contradiction that $\nabla x = \nabla y$.

By assumption $\exists w \in \Sigma^*$ such that $\nabla x w \in A$ and $\nabla y w \notin A$. 
Distinguishable strings means different states: Proof

**Lemma**

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By assumption $\exists w \in \Sigma^*$ such that $\nabla xw \in A$ and $\nabla yw \notin A$.

$$\implies A \ni \nabla xw = \delta^*(s, xw) = \delta^*(\nabla x, w)$$
Distinguishable strings means different states: Proof

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$\implies A \ni \nabla yw \notin A$. Impossible!
**Lemma**

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$\implies A \ni \nabla yw \notin A$. Impossible!

Assumption that $\nabla x = \nabla y$ is false. \qed
• Prove for any $i \neq j$ then $0^i$ and $0^j$ are distinguishable for the language $\{0^n1^n \mid n \geq 0\}$.

$0^i$ and $0^j$ are distinguishable.

We need to show that $\exists x \in \Sigma^*$ such that

\[ 0^ix \in L \text{ but } 0^jx \notin L \quad i \neq j \]

Let $x = 1^i$.

$\Rightarrow 0^i1 \in L \quad 0^j1 \notin L$

$\Rightarrow 0^i$ and $0^j$ are distinguishable!
Review questions...

- Prove for any $i \neq j$ then $0^i$ and $0^j$ are distinguishable for the language $\{0^n1^n \mid n \geq 0\}$.

- Let $L$ be a regular language, and let $w_1, \ldots, w_k$ be strings that are all pairwise distinguishable for $L$. Prove any DFA for $L$ must have at least $k$ states.
Review questions...

- Prove for any $i \neq j$ then $0^i$ and $0^j$ are distinguishable for the language $\{0^n1^n \mid n \geq 0\}$.

- Let $L$ be a regular language, and let $w_1, \ldots, w_k$ be strings that are all pairwise distinguishable for $L$. Prove any DFA for $L$ must have at least $k$ states.

- Prove that $\{0^n1^n \mid n \geq 0\}$ is not regular.

\begin{itemize}
  \item By way of contradiction, let $L = \{0^n1^n \mid n \geq 0\}$ be regular.
  \item Therefore, there exists a DFA $M = (Q, \Sigma, \delta, s_0, A)$ with $|Q|$ finite.
  \item Then $0^i$ and $0^j$ are distinguishable for $L$ ! ($\Leftarrow$)
  \item $i \neq j$ and $0$, $00$, $000$, $\ldots$ are infinitely many distinguishable prefixes.
  \item Therefore, $|Q| = \infty \Rightarrow L$ is not regular.
\end{itemize}
Fooling sets: Proving non-regularity
Fooling Sets

Definition
For a language $L$ over $\Sigma$ a set of strings $F$ (could be infinite) is a fooling set or distinguishing set for $L$ if every two distinct strings $x, y \in F$ are distinguishable.

Example:
$F = \{0^i \mid i \geq 0\}$ is an infinite fooling set for $L$!
Fooling Sets

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For a language $L$ over $\Sigma$ a set of strings $F$ (could be infinite) is a fooling set or distinguishing set for $L$ if every two distinct strings $x, y \in F$ are distinguishable.

Example: $F = \{0^i \mid i \geq 0\}$ is a fooling set for the language $L = \{0^n1^n \mid n \geq 0\}$.
Fooling Sets

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For a language $L$ over $\Sigma$ a set of strings $F$ (could be infinite) is a fooling set or distinguishing set for $L$ if every two distinct strings $x, y \in F$ are distinguishable.

Example: $F = \{0^i \mid i \geq 0\}$ is a fooling set for the language $L = \{0^n1^n \mid n \geq 0\}$.

Theorem
Suppose $F$ is a fooling set for $L$. If $F$ is finite then there is no DFA $M$ that accepts $L$ with less than $|F|$ states.
Already proved the following lemma:

**Lemma**  
$L$: regular language.

$M = (Q, \Sigma, \delta, s, A)$: **DFA** for $L$.

*If* $x, y \in \Sigma^*$ *are distinguishable*, then $\nabla x \neq \nabla y$.

Reminder: $\nabla x = \delta^*(s, x)$.  
Theorem (Reworded.)

$L$: A language

$F$: a fooling set for $L$.

If $F$ is finite then any DFA $M$ that accepts $L$ has at least $|F|$ states.

Proof.
Let $F = \{w_1, w_2, \ldots, w_m\}$ be the fooling set.

Let $M = (Q, \Sigma, \delta, s, A)$ be any DFA that accepts $L$. 

Theorem (Reworded.)

$L$: A language

$F$: a fooling set for $L$.

If $F$ is finite then any DFA $M$ that accepts $L$ has at least $|F|$ states.

Proof.

Let $F = \{w_1, w_2, \ldots, w_m\}$ be the fooling set.

Let $M = (Q, \Sigma, \delta, s, A)$ be any DFA that accepts $L$.

Let $q_i = \nabla w_i = \delta^*(s, x_i)$. 
Proof of theorem

**Theorem (Reworded.)**

$L$: A language

$F$: a fooling set for $L$.

*If $F$ is finite then any DFA $M$ that accepts $L$ has at least $|F|$ states.*

**Proof.**

Let $F = \{w_1, w_2, \ldots, w_m\}$ be the fooling set.

Let $M = (Q, \Sigma, \delta, s, A)$ be any DFA that accepts $L$.

Let $q_i = \nabla w_i = \delta^*(s, x_i)$.

By lemma $q_i \neq q_j$ for all $i \neq j$.

As such, $|Q| \geq |\{q_1, \ldots, q_m\}| = |\{w_1, \ldots, w_m\}| = |A|$. \qed
Corollary
If $L$ has an infinite fooling set $F$ then $L$ is not regular.

Proof.
Let $w_1, w_2, \ldots \subseteq F$ be an infinite sequence of strings such that every pair of them are distinguishable.

Assume for contradiction that $\exists$ $M$ a DFA for $L$. 
Corollary
If $L$ has an infinite fooling set $F$ then $L$ is not regular.

Proof.
Let $w_1, w_2, \ldots \subseteq F$ be an infinite sequence of strings such that every pair of them are distinguishable.

Assume for contradiction that $\exists M$ a DFA for $L$.

Let $F_i = \{w_1, \ldots, w_i\}$.

By theorem, $\#$ states of $M \geq |F_i| = i$, for all $i$.

As such, number of states in $M$ is infinite.
Corollary
If $L$ has an infinite fooling set $F$ then $L$ is not regular.

Proof.
Let $w_1, w_2, \ldots \subseteq F$ be an infinite sequence of strings such that every pair of them are distinguishable.

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As such, number of states in $M$ is infinite.

Contradiction: DFA = deterministic finite automata. But $M$ not finite.
Examples

\[ L_1 \cdot \{0^n1^n \mid n \geq 0\} \quad F = \{0^i1^i \mid i \geq 0\} \]

Non-regular!

\[ L_2 \cdot \{\text{bitstrings with equal number of 0s and 1s}\} \]

\[ L_1 \subset L_2 \]

\[ L_3 \cdot \{0^k1^\ell \mid k \neq \ell\} \]

Check if the same
F works?

\[ \begin{align*}
01 & \in L_1 \\
01 & \in L_2 \\
10 & \in L_2 \\
10 & \notin L_1
\end{align*} \]

(\text{True}?)
$L = \{\text{strings of properly matched open and closing parentheses}\}$

Regular or not?

$F = \{c^k \mid k \geq 0\}$
Examples

\[ L = \{ \text{palindromes over the binary alphabet } \Sigma = \{0, 1\} \} \]

A palindrome is a string that is equal to its reversal, e.g. 10001 or 0110.

\[ F = \{ (01)^i \mid i \geq 0 \} \]

\[ (01)^i (10)^i \in L \]

\[ (01)^j (10)^i \notin L \]

\[ x = (10)^i \]

\[ |F| = \infty \]

\[ i = 1 \quad j = 2 \]

\[ 0110 \in L \]

\[ 010110 \notin L \]
Closure properties: Proving non-regularity
Non-regularity via closure properties

\[ H = \{ \text{bitstrings with equal number of 0s and 1s} \} \]

\[ H' = \{ 0^k1^k \mid k \geq 0 \} \]

Suppose we have already shown that \( H' \) is non-regular. Can we show that \( H \) is non-regular without using the fooling set argument from scratch?

Suppose \( H \) is regular. Then since \( L(0^*1^*) \) is regular, and regular languages are closed under intersection, \( H_0 \) also would be regular. But we know \( H_0 \) is not regular, a contradiction.

Hence, \( H \) is non-regular.
Non-regularity via closure properties

\[ H = \{ \text{bitstrings with equal number of 0s and 1s} \} \]

\[ H' = \{ 0^k1^k \mid k \geq 0 \} \]

Suppose we have already shown that \( L' \) is non-regular. Can we show that \( L \) is non-regular without using the fooling set argument from scratch?

\[ H' = H \cap L(0^*1^*) \]

Claim: The above and the fact that \( L' \) is non-regular implies \( L \) is non-regular. Why?

(We covered it in the last slide!)
Non-regularity via closure properties

\[ H = \{ \text{bitstrings with equal number of 0s and 1s} \} \]

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Suppose we have already shown that \( L' \) is non-regular. Can we show that \( L \) is non-regular without using the fooling set argument from scratch?

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Suppose \( H \) is regular. Then since \( L(0^*1^*) \) is regular, and regular languages are closed under intersection, \( H' \) also would be regular. But we know \( H' \) is not regular, a contradiction.
Non-regularity via closure properties

General recipe:

L₁, L₂, ..., Lₙ

Apply closure properties

E.g.: {ε}, {00, 10}, (0+1)*, 0*1*, ...

KNOWN REGULAR

UNKNOWN

L non-regular

KNOWN
Examples

\[ L = \{0^k1^k \mid k \geq 1\} \]

Non-Reg!

\[ L_1 = \{0^k1^k \mid k \geq 0\} \text{ is non-reg.} \]

\[ L_1 = L \cup \{e\} \text{ is reg.} \]

\[ \text{non-reg.} \]

\[ \text{non-reg.} \]
$L' = \{0^k1^k \mid k \geq 0\}$

Complement of $L$ ($\overline{L}$) is also not regular.

But $L \cup \overline{L} = (0 + 1)^*$ which is regular.

In general, always use closure in forward direction, i.e., $L$ and $L'$ are regular, therefore $L \cup L'$ is regular.

In particular, regular languages are not closed under subset/superset relations.
Proving non-regularity: Summary

• **Method of distinguishing suffixes.** To prove that $L$ is non-regular find an infinite fooling set.

• **Closure properties.** Use existing non-regular languages and regular languages to prove that some new language is non-regular.

• **Pumping lemma.** We did not cover it but it is sometimes an easier proof technique to apply, but not as general as the fooling set technique.